

Scaling in erosion of landscapes: Renormalization group analysis of a model with infinitely many couplings

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Abstract. Standard field theoretic renormalization group is applied to the model of landscape erosion introduced by R. Pastor-Satorras and D. H. Rothman [*Phys. Rev. Lett.* **80**: 4349 (1998); *J. Stat. Phys.* **93**: 477 (1998)] yielding unexpected results: the model is multiplicatively renormalizable only if it involves infinitely many coupling constants, (*i.e.*, the corresponding renormalization group equations involve infinitely many β -functions). Despite this fact, the one-loop counterterm can be derived albeit in a closed form in terms of the certain function $V(h)$, entering the original stochastic equation, and its derivatives with respect to the height field h . Its Taylor expansion gives rise to the full infinite set of the one-loop renormalization constants, β -functions and anomalous dimensions. Instead of a set of fixed points, there is a two-dimensional surface of fixed points that is likely to contain infrared attractive region(s). If that is the case, the model exhibits scaling behaviour in the infrared range. The corresponding critical exponents are nonuniversal through the dependence on the coordinates of the fixed point on the surface, but satisfy certain universal exact relations.

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1. Introduction and description of the model

Over decades, constant interest has been attracted to the problem of landscape erosion due to the flow of air or water over it, and to related problems like, *e.g.* granular flows; see Refs. [1]–[18] and the literature cited therein. Of course, those issues concern a wide variety of diverse physical phenomena; the underlying dynamical models have been a source of much controversy [4]–[17]. However, in analogy with critical phenomena, one can hope that universal aspects of landscape erosion (like the exponents in scaling laws) can be described within the framework of relatively simple semiphenomenological models, constructed on the basis of dimensionality and symmetry considerations; see, *e.g.* the discussion in [14, 15] and references therein.

Similar situation takes place in the related problem of kinetic roughening of surfaces or interfaces, described by the well known Kardar-Parisi-Zhang stochastic model [19] and its descendants [20]–[22]. Another example is provided by the problem of self-organized

criticality, which in the continuum limit is described by the Hwa-Kardar stochastic model [23] and its modifications [24, 25].

For the erosion of a surface with a fixed mean tilt, analogous model was proposed in [14, 15]. Let us describe that model first.

Let \mathbf{n} be a unit constant vector that determines a certain preferred direction (direction of the slope) and, therefore, introduces intrinsic anisotropy into the model. Then any vector can be decomposed into the components perpendicular and parallel to \mathbf{n} . In particular, for the d -dimensional horizontal position \mathbf{x} one has $\mathbf{x} = \mathbf{x}_\perp + \mathbf{n}x_\parallel$ with $\mathbf{x}_\perp \cdot \mathbf{n} = 0$. In the following, we denote the derivative in the full d -dimensional \mathbf{x} space by $\partial = \partial/\partial x_i$ with $i = 1 \dots d$, and the derivative in the subspace orthogonal to \mathbf{n} by $\partial_\perp = \partial/\partial x_{\perp i}$ with $i = 1 \dots d-1$. Then the derivative in the parallel direction is written as $\partial_\parallel = \mathbf{n} \cdot \partial$.

The stochastic differential equation for the height of the profile, *i.e.* for the height field $h(\mathbf{x}) = h(t, \mathbf{x})$, proposed in [14, 15] is taken in the form

$$\partial_t h = \nu_\perp \partial_\perp^2 h + \nu_\parallel \partial_\parallel^2 h + \partial_\parallel^2 V(h) + f. \quad (1.1)$$

Here $\partial_t = \partial/\partial t$, ν_\parallel and ν_\perp are topographic diffusion coefficients, $V(h)$ is some function that depends only on the field $h(\mathbf{x})$ (and not on its derivatives) and $f(\mathbf{x})$ is a Gaussian random noise with zero mean and prescribed pair correlation function

$$\langle f(\mathbf{x}) f(\mathbf{x}') \rangle = D \delta(t - t') \delta^{(d)}(\mathbf{x} - \mathbf{x}') \quad (1.2)$$

with some positive amplitude D . Detailed discussion of the derivation of the model (1.1), (1.2) and its relationship to other models of erosion and self-organized criticality is given in [14, 15].

The function $V(h)$ understood as series in powers of h . In [14, 15] it was taken odd in h : this is dictated by the symmetry $h, f \rightarrow -h, -f$; another symmetry of the model is $x_\parallel \rightarrow -x_\parallel$. The authors of [14, 15] truncated the Taylor expansion of $V(h)$ on the leading h^3 term (the term linear in h is written in (1.1) separately) and then applied to the resulting model the dynamic Wilsonian renormalization group (RG) and the expansion in $4 - d$, the deviation of the dimension d from its supposed upper critical value $d = 4$. In the leading one-loop order, they established existence of the infrared (IR) attractive fixed point and calculated the corresponding critical (roughness) exponents in a good agreement with the experimental data obtained from sea floor measurements.

In the present paper we apply to the model [14, 15] the standard field theoretic RG and arrived at completely different results. The plan of the paper and the main results are as follows.

In section 2 we present the field theoretic formulation of the stochastic problem (1.1), (1.2) for the arbitrary (not necessarily odd) full-scale (not truncated) function $V(h)$.

In section 3 we discuss ultraviolet (UV) divergences and renormalization procedure of the resulting field theory. We show that the upper critical dimension is in fact $d = 2$. This leads to drastic change in the RG analysis of the model. Namely, the higher-order terms of the Taylor expansion of $V(h)$ cannot be dropped, because they unavoidably

appear as counterterms in the correct renormalization procedure. In other words, any truncated model is not multiplicatively renormalizable. This means that the properly constructed renormalized model necessarily involves infinitely many coupling constants, and the corresponding RG equations involve infinitely many β -functions. This also means that the RG analysis performed in Refs. [14, 15] for the truncated model is not self-consistent and its results cannot be considered reliable.

We write down the corresponding renormalized action functional, renormalization relations for the fields and parameters, RG equations and RG functions (β -functions and anomalous dimensions).

In section 4 we explicitly perform the renormalization in the leading one-loop order. The key point is that, despite the fact that the model involves infinitely many couplings, the one-loop counterterm can be derived in a closed form in terms of the function $V(h)$ and its derivatives. Its Taylor expansion gives rise to the full infinite set of one-loop renormalization constants, and, therefore, to all β -functions and anomalous dimensions.

In this derivation, we adopt the functional method applied earlier by A. N. Vasil'ev and one of the authors [27] to an isotropic model of surface roughening, proposed in [26] as a possible modification of the Kardar-Parisi-Zhang equation; see also [28, 29].

In section 5 we analyze attractors of the obtained RG equations in the infinite-dimensional space of coupling constants. It turns out, that instead of a set of fixed points (like for most multicoupling models), there is a two-dimensional surface of fixed points. For odd $V(h)$, that is, for the model [14, 15], it reduces to a curve. It seems likely that it contains IR attractive region(s). If so, the model exhibits scaling behaviour in the IR range. The corresponding critical exponents are nonuniversal through the dependence of the coordinates of the fixed point on the surface (curve), but satisfy certain exact relations.

Possible consequences for the comparison with the experiments and remaining problems are briefly discussed in section 6.

2. Field Theoretic Formulation of the Model

According to the general statement (see, *e.g.* the books [30, 31] and the references therein), the stochastic problem (1.1), (1.2) is equivalent to the field theoretic model of the doubled set of fields $\Phi = \{h, h'\}$ with the action functional

$$\mathcal{S}(\Phi) = h'h' + h' \left\{ -\partial_t h + \nu_{0\perp} \partial_{\perp 0}^2 h + \nu_{\parallel 0} \partial_{\parallel}^2 h + \partial_{\parallel}^2 \sum_{n=2}^{\infty} \frac{\lambda_{n0} h^n}{n!} \right\} \quad (2.1)$$

(we have scaled out D_0 and other factors of $h'h'$ by adjusting the values of λ_{n0}). Here and below, all the needed integrations over $x = (t, \mathbf{x})$ and summations over repeated tensor indices are always implied, *e.g.*

$$h'h' = \int dt \int d\mathbf{x} h'(t, \mathbf{x}) h'(t, \mathbf{x}). \quad (2.2)$$

The subscript 0 means that the parameters in (2.1) are not yet renormalized (bare).

The field theoretic formulation means that various correlation and response functions of the stochastic problem (1.1), (1.2) can be identified with various Green's functions of the field theoretic model with the action (2.1). In other words, they are represented by functional averages over the full set of fields $\Phi = \{h, h'\}$ with the weight $\exp \mathcal{S}(\Phi)$.

3. UV divergences and renormalization

The analysis of canonical dimensions is employed to analyze the UV divergences; see, *e.g.* [30, 31]. Conventional dynamic models of the type (2.1) have two scales, and their dimensions are described by the two numbers - the frequency dimension d_F^ω , and the momentum dimension d_F^k . They completely define the canonical dimension of a quantity F (a field or a parameter), and are determined so that $[F] \sim [T]^{-d_F^\omega} [L]^{-d_F^k}$, where L is the typical length scale and T is the time scale; see, *e.g.* Chap. 5 in book [31]. In the present case, however, due to the anisotropy there are two independent momentum scales, related to the directions perpendicular and parallel to the vector \mathbf{n} which requires a more detailed specification. Namely, two independent momentum canonical dimensions d_F^\perp and d_F^\parallel had to be introduced so that

$$[F] \sim [T]^{-d_F^\omega} [L_\perp]^{-d_F^\perp} [L_\parallel]^{-d_F^\parallel},$$

where L_\perp and L_\parallel are (independent) length scales in the corresponding subspaces. The obvious normalization conditions are $d_{k_\perp}^\perp = -d_{\mathbf{x}_\perp}^\perp = 1$, $d_{k_\perp}^\parallel = -d_{\mathbf{x}_\perp}^\parallel = 0$, $d_{k_\perp}^\omega = d_{k_\parallel}^\omega = 0$, $d_\omega^\omega = -d_t^\omega = 1$, *etc.*; the requirement that each term of the action functional (2.1) be dimensionless (with respect to all the three independent dimensions separately) is the last condition needed to find the dimensions. The original momentum dimension can be found from the relation $d_F^k = d_F^\perp + d_F^\parallel$. Then, based on d_F^k and d_F^ω , the total canonical dimension can be introduced $d_F = d_F^k + 2d_F^\omega = d_F^\perp + d_F^\parallel + 2d_F^\omega$ (in the free theory, $\partial_t \propto \partial_\perp^2 \propto \partial_\parallel^2$), which plays in the theory of renormalization of dynamic models the same role as the conventional (momentum) dimension does in static problems; see, *e.g.* Chap. 5 in book [31].

The canonical dimensions for the model (2.1) are presented in table 1. The renormalized parameters (without the subscript 0) and the renormalization mass μ will be introduced later.

Table 1. Canonical dimensions of the fields and the parameters in the model (2.1)

F	h'	h	ν_\perp	ν_\parallel	λ_{n0}	g_{n0}	g_n	μ
d_F^ω	1/2	-1/2	1	1	$(n+1)/2$	0	0	0
d_F^\parallel	1/2	1/2	0	-2	$-(n+3)/2$	0	0	0
d_F^\perp	$(d-1)/2$	$(d-1)/2$	-2	0	$(d-1)(1-n)/2$	$(2-d)(n-1)/2$	0	1
d_F	$d/2+1$	$-(2-d)/2$	0	0	$(2-d)(n-1)/2$	$(2-d)(n-1)/2$	0	1

From table 1 we see that all the coupling constants g_{n0} become simultaneously dimensionless at $d = 2$. This means that $d = 2$ is the upper critical dimension for the full-scale model. For this value of d , the total canonical dimension of the field h vanishes. As explained below, this fact leads to serious consequences for the renormalization procedure. This fact also means that UV divergences in the Green's functions of the full-scale model manifest themselves as poles in $\varepsilon = 2 - d$, and that ε plays the role of the expansion parameter in the RG expansions.

The total canonical dimension of an arbitrary 1-irreducible Green's function $\Gamma = \langle \Phi \cdots \Phi \rangle_{1\text{-ir}}$ with $\Phi = \{h, h'\}$ in the frequency-momentum representation is given by the relation:

$$d_\Gamma = d + 2 - d_h N_h - d_{h'} N_{h'}, \quad (3.1)$$

where $N_h, N_{h'}$ are the numbers of the corresponding fields entering into the function Γ ; see, *e.g.* [31].

The total dimension d_Γ in the logarithmic theory (*i.e.* at $\varepsilon = 0$) is, in fact, the formal index of the UV divergence: $\delta_\Gamma = d_\Gamma|_{\varepsilon=0}$. The superficial UV divergences, whose removal requires counterterms, can be present only in those functions Γ for which δ_Γ is a non-negative integer. The counterterm is a polynomial in frequencies and momenta of degree δ_Γ (given that the convention that $\omega \propto k^2$ is implied).

If a number of external momenta occurs as an overall factor in all diagrams of a certain Green's function, the real index of divergence δ'_Γ will be smaller than δ_Γ by the corresponding number. This is exactly what happens in our model: using integration by parts, the derivative at the vertex $h' \partial_\parallel^2 V(h)$ can be moved onto the field h' . This means that any appearance of h' in some function Γ gives a square of such an external momentum, and the real index of divergence is given by the expression $\delta'_\Gamma = \delta_\Gamma - 2N_{h'}$. Moreover, h' can appear in the corresponding counterterm only in the form of derivative.

From table 1 and the expression (3.1) one obtains:

$$\delta'_\Gamma = \delta_\Gamma - 2N_{h'} = 4 - 4N_{h'}. \quad (3.2)$$

It is sufficient to consider only the case $N_{h'} > 0$ because all the 1-irreducible Green's functions without the response fields vanish identically in dynamical models (their diagrams always involve closed circuits of retarded lines); see, *e.g.* [31].

Straightforward analysis of the expression (3.2) shows that superficial UV divergences can be present only in the 1-irreducible functions of the form $\langle h'h \dots h \rangle_{1\text{-ir}}$ with the counter-term $(\partial_\parallel^2 h')h^n$ (for any $n \geq 1$). Indeed, all the other counter-terms (*e.g.* $h'h'$, $h'\partial_t h$, $h'\partial_\perp^2 h$) are not needed as the corresponding 1-irreducible functions are finite.

As all the terms $(\partial_\parallel^2 h')h^n$ are present in the action (2.1), the full model is multiplicatively renormalizable. The renormalized action can be written in the form:

$$\mathcal{S}_R(\Phi) = h'h' + h' \left\{ -\partial_t h + Z_\perp \nu_\perp \partial_\perp^2 h + Z_\parallel \nu_\parallel \partial_\parallel^2 h + \partial_\parallel^2 \sum_{n=2}^{\infty} \frac{Z_n \lambda_n h^n}{n!} \right\}. \quad (3.3)$$

Here ν_\perp , ν_\parallel and λ_n are renormalized analogs of the bare parameters (those with subscript 0). The renormalization constants Z_\perp , Z_\parallel , and Z_n depend only on the completely dimensionless parameters g_n and absorb the poles in ε . The bare charges $g_0 = \{g_{n0}\}$ and completely dimensionless renormalized charges $g = \{g_n\}$ ($n = 2, 3, \dots$) are expressed in terms of bare parameters λ_{n0} and renormalized parameters λ_n as follows:

$$\lambda_{n0} = g_{n0} \nu_{\parallel 0}^{(n+3)/4} \nu_{\perp 0}^{(n-1)/4}, \quad \lambda_n = g_n \nu_\parallel^{(n+3)/4} \nu_\perp^{(n-1)/4} \mu^{\varepsilon(n-1)/2}, \quad (3.4)$$

Here the renormalization mass μ is an additional parameter of the renormalized theory; its canonical dimensions are shown in table 1.

The renormalized action (3.3) is obtained from the original one (2.1) by the renormalization of the parametrs (the renormalization of the fields h, h' is not required):

$$\nu_{\parallel 0} = \nu_\parallel Z_\parallel, \quad \nu_{\perp 0} = \nu_\perp Z_\perp, \quad g_{n0} = \mu^{\varepsilon(n-1)/2} g_n Z_{g_n}, \quad \lambda_{n0} = \lambda_n Z_n. \quad (3.5)$$

The renormalization constants in Eqs. (3.3) and (3.5) are related as follows:

$$Z_{g_n} = Z_n Z_\parallel^{-(n+3)/4} Z_\perp^{-(n-1)/4}. \quad (3.6)$$

Let us consider an elementary derivation of the RG equations [30, 31]. The RG equations are written for the renormalized Green's functions $G_R = \langle \Phi \cdots \Phi \rangle_R$. In the present case they are equal to the original (unrenormalized) Green's functions G : $G(e_0, \dots) = G_R(e, \mu, \dots)$ (because there is no renormalization for the fields) and, therefore, can be equally used for analyzing the critical behaviour. Here, $e_0 = \{g_{n0}, \nu_{\parallel 0}, \nu_{\perp 0}, \dots\}$ is a full set of bare parameters and $e = \{g_n, \nu_\parallel, \nu_\perp, \dots\}$ are their renormalized counterparts; the ellipsis stands for the other arguments (times, coordinates, momenta etc.).

We use $\tilde{\mathcal{D}}_\mu$ to denote the differential operation $\mu \partial_\mu|_{e_0}$. When expressed in the renormalized variables it looks as follows:

$$\mathcal{D}_{RG} \equiv \mathcal{D}_\mu + \sum_{n=2}^{\infty} \beta_n \partial_{g_n} - \sum_{F=\nu_\parallel, \nu_\perp} \gamma_F \mathcal{D}_F, \quad (3.7)$$

where $\mathcal{D}_x \equiv x \partial_x$ for any variable x . The anomalous dimensions γ are defined as

$$\gamma_F \equiv \tilde{\mathcal{D}}_\mu \ln Z_F \quad \text{for any quantity } F, \quad (3.8)$$

and the β functions for the dimensionless coupling constants g_n are

$$\beta_n \equiv \tilde{\mathcal{D}}_\mu g_n = g_n [-\varepsilon(n-1)/2 - \gamma_{g_n}]. \quad (3.9)$$

4. One-loop expressions for the counterterm, renormalization constants and RG functions

Let us turn to the calculation of the constants Z in the one-loop approximation. Despite the fact that the full renormalizable model involves infinitely many coupling constants, the one-loop counterterm can be calculated in an explicit closed form in terms of the function $V(h)$.

Consider the expansion of the generating functional $\Gamma_R(\Phi)$ of the 1-irreducible Green's functions of our model in the number p of loops:

$$\Gamma_R(\Phi) = \sum_{p=0}^{\infty} \Gamma^{(p)}(\Phi), \quad \Gamma^{(0)}(\Phi) = S_R(\Phi). \quad (4.1)$$

The loopless (tree-like) contribution is simply the action while the one-loop contribution can be calculated via following relation, see, *e.g.* [32]:

$$\Gamma^{(1)}(\Phi) = -(1/2)\text{Tr} \ln(W/W_0), \quad (4.2)$$

where W is a linear operation with the kernel

$$W(x, y) = -\delta^2 S_R(\Phi) / \delta\Phi(x) \delta\Phi(y), \quad (4.3)$$

and W_0 is the similar expression for the free parts of the action. The both W and W_0 are 2×2 -matrices in the pair $\Phi = \{h, h'\}$.

The requirement that UV divergences in (4.1) are removed, along with the minimal subtraction prescription, provides the uniquely determined values for constants Z . In the one-loop approximation we put $Z = 1$ in (4.2) while keeping leading-order terms in the coupling constants g_n in the loopless contribution in the constants Z ; for internal consistency we suppose that $g_n \simeq g_2^{n-2}$.

Let us represent the Taylor expansion of the function $V(h)$ as follows:

$$V(h) = \sum_{n=2}^{\infty} \lambda_n h^n(x) / n!, \quad V_R(h) = \sum_{n=2}^{\infty} Z_n \lambda_n h^n(x) / n!, \quad (4.4)$$

In the following, we interpret similar objects as functions of a single variable $h(x)$, and V' , V'' , etc., as the corresponding derivatives with respect to this variable. In this notation the matrix W (under condition that $Z = 1$) can be symbolically represented as

$$W = \begin{pmatrix} -\partial_{\parallel}^2 h' \cdot V'' & L^T \\ L & -2 \end{pmatrix}, \quad (4.5)$$

where $L \equiv \partial_t - \nu_{\parallel} \partial_{\parallel}^2 - \nu_{\perp} \partial_{\perp}^2 - \partial_{\parallel}^2 V'$, and $L^T \equiv -\partial_t - \nu_{\parallel} \partial_{\parallel}^2 - \nu_{\perp} \partial_{\perp}^2 - V' \partial_{\parallel}^2$ is the transposed operation.

In order to calculate the constants Z we need only the divergent part of expression (4.1), which was previously established to have the form

$$\int dx \partial^2 h'(x) R(h(x))$$

with a function $R(h)$ similar to $V(h)$. This means that we need to calculate $\text{Tr} \ln$ in (4.2) with matrix (4.5) only to the first order in its hh -element $-\partial_{\parallel}^2 h' \cdot V''$. We can do this employing the well-known formula $\delta(\text{Tr} \ln K) = \text{Tr}(K^{-1} \delta K)$ for any variation δK . By varying only the hh -element of the matrix W we obtain

$$\begin{aligned} \int dx \partial^2 h'(x) R(h(x)) &\simeq -\text{Tr} [D_{hh} V'' \partial_{\parallel}^2 h'] = \\ &= - \int dx D^{(hh)}(x, x) V''(h(x)) \partial_{\parallel}^2 h'(x), \end{aligned} \quad (4.6)$$

where $D^{hh} = (W^{-1})_{hh}$ at $h' = 0$. By the definition, D^{hh} is the ordinary propagator $\langle hh \rangle$ of the model (3.3) with $Z = 1$ and with $\nu_{\parallel} \partial_{\parallel}^2 + \nu_{\perp} \partial_{\perp}^2 + \partial_{\parallel}^2 V'$ substituted for $\nu_{\parallel} \partial_{\parallel}^2 + \nu_{\perp} \partial_{\perp}^2$.

There is another consideration that must be taken into account. After ∂_{\parallel}^2 is moved to the external factor h' only a logarithmically divergent expression remains in the counterterm. This means that we can set all its external momenta to zero while calculating the divergent part of a given diagram (IR regularization is ensured by the cutoff). In its turn, this leads to the fact that we can ignore the inhomogeneity of $\partial_{\parallel}^2 h'(x)$ and $h(x)$ (both can be assumed to be constant) in (4.6) when we select the poles in ε . Then $D_{hh}(x, x)$ can easily be calculated by going over to the momentum-frequency representation:

$$\begin{aligned} D_{hh}(x, x) &= \int \int \frac{d\omega d\mathbf{k}}{(2\pi)^{d+1}} \frac{2}{\omega^2 + [\nu_{\parallel} k_{\parallel}^2 + \nu_{\perp} k_{\perp}^2 + k_{\parallel}^2 V']^2} = \\ &= \frac{S_d}{(2\pi)^d} \frac{\mu^{-\varepsilon}}{\varepsilon} \frac{1}{\sqrt{\nu_{\perp}(\nu_{\parallel} + V')}} + \dots, \end{aligned} \quad (4.7)$$

where the elipsis stands for the UV-finite part.

Substituting (4.6) and (4.7) into (4.2) yields the following expression for the divergent part of $\Gamma_1(\Phi)$ with the required accuracy:

$$\Gamma_1(\Phi) = \frac{S_d}{2(2\pi)^d} \frac{\mu^{-\varepsilon}}{\varepsilon} \int dx \frac{V''(h(x))}{\sqrt{\nu_{\perp}(\nu_{\parallel} + V'(h(x)))}} \partial^2 h'(x) \quad (4.8)$$

We can find the one-loop contributions of order $1/\varepsilon$ in all constants Z due to the fact that the sum of (4.8) and the loopless contribution in (4.2) has no pole in ε (it cancels out).

Let us introduce the representation

$$\frac{V''(h(x))}{\sqrt{\nu_{\perp}(\nu_{\parallel} + V'(h(x)))}} = \sum_{n=0}^{\infty} \mu^{\varepsilon(n+1)/2} \nu_{\perp}^{(n-1)/4} \nu_{\parallel}^{(n+3)/4} \frac{r_n h^n}{n!}, \quad (4.9)$$

for the Taylor expansion of the integrand in (4.8).

Then r_n are completely dimensionless coefficients – polynomials in the charges g_n . Combining the above condition for the canceling out of poles in ε and (3.4), we get

$$Z_{\perp} = 1, \quad Z_{\parallel} = 1 - \frac{r_1 S_d}{2(2\pi)^d \varepsilon} + \dots \quad Z_n = 1 - \frac{r_n}{g_n} \frac{S_d}{2(2\pi)^d \varepsilon} + \dots \quad (4.10)$$

The operation $\tilde{\mathcal{D}}_{\mu}$ in (3.9) assumes the form

$$\tilde{\mathcal{D}}_{\mu} = \sum_n \left(\tilde{\mathcal{D}}_{\mu} g_n \right) \partial_{g_n} = \sum_n \beta_n \partial_{g_n}.$$

So in order to achieve the required accuracy it is sufficient to use only the first terms in the β -functions (3.9). This yields

$$\tilde{\mathcal{D}}_{\mu} \simeq -\frac{\varepsilon}{2} \mathcal{D}_g, \quad \mathcal{D}_g = \sum_{n=2}^{\infty} (n-1) g_n \partial_{g_n}. \quad (4.11)$$

This consideration together with (4.10), (3.6), and (3.9) leads to the following expressions for the one-loop RG-functions:

$$\gamma_{\parallel} = a\mathcal{D}_g r_1/2, \quad a \equiv \frac{S_d}{2(2\pi)^d}; \quad (4.12)$$

$$\beta_n = -\varepsilon \frac{n-1}{2} g_n + \frac{n+3}{4} g_n \gamma_{\parallel} - \frac{a}{2} (\mathcal{D}_g - n + 1) r_n. \quad (4.13)$$

The explicit expressions for the first four coefficients r_n [the first term with r_0 in (4.9) contributes nothing to (4.8)] are found from the definitions (4.9), (4.4), (3.4):

$$\begin{aligned} r_1 &= g_3 - \frac{1}{2} g_2^2, & r_2 &= g_4 - \frac{3}{2} g_2 g_3 + \frac{3}{4} g_2^3, \\ r_3 &= g_5 - 2g_2 g_4 - \frac{3}{2} g_3^2 + \frac{9}{2} g_2^2 g_3 - \frac{15}{8} g_2^4, \\ r_4 &= g_6 - \frac{5}{2} g_2 g_5 + \frac{15}{2} g_2^2 g_4 - 5g_3 g_4 + \frac{45}{4} g_2 g_3^2 - \frac{75}{4} g_2^3 g_3 + \frac{105}{16} g_2^5, \end{aligned}$$

when substituted into (4.12) they yield:

$$\gamma_{\parallel} = \frac{a}{2} (2g_3 - g_2^2), \quad (4.14)$$

$$\begin{aligned} \beta_2 &= -\frac{\varepsilon}{2} g_2 + a \left(-g_4 + \frac{11}{4} g_2 g_3 - \frac{1}{8} g_2^3 \right), \\ \beta_3 &= -\varepsilon g_3 + a \left(-g_5 + 2g_2 g_4 + 3g_3^2 - \frac{21}{4} g_2^2 g_3 + \frac{15}{8} g_2^4 \right) \end{aligned} \quad (4.15)$$

We recall that we have to admit $g_n \sim g_2^{(n-1)}$ for the sake of consistency of the approximation.

5. Attractors and critical exponents

Let us turn to the complete system (4.15) of the β -functions. The fixed points of RG equations can be found from the requirement that $\beta_n(g_*) = 0$, $n = 2, 3, \dots$. The explicit form of the β -functions (4.15) shows that we can choose the coordinates g_{2*} , and g_{3*} arbitrarily, while all the other g_{n*} with $n \geq 4$ are then uniquely determined from the equations $\beta_k(g_*) = 0$, $k \geq 3$. This means that in the infinite-dimensional space of the couplings $g \equiv \{g_n\}$ the RG-equation (3.7) has a two-dimensional surface of fixed points, parametrized by the values of g_{2*} , and g_{3*} .

In general case, studying these points is a difficult task. However, according to the general rule [30], a point g_* is IR stable if the real parts of all the eigen-numbers of the matrix $\omega_{nm} = \partial\beta_n/\partial g_m|_{g_*}$ are strictly positive. The requirement that all the diagonal elements ω_{nn} be positive is the necessary condition for IR-stability. Equation (4.13) can be used to calculate these elements for all values of n :

$$\omega_{22} = -\frac{\varepsilon}{2} + a \left[\frac{11}{4} g_{3*} - \frac{3}{8} g_{2*}^2 \right], \quad \omega_{33} = -\varepsilon + a \left[6g_{3*} - \frac{21}{4} g_{2*}^2 \right],$$

and for $n \geq 4$ we have

$$\omega_{nn} = -\varepsilon \frac{n-1}{2} + a \frac{(n+1)^2 + 2}{4} g_{3*} - \frac{a}{8} (n(3n+4) + 3) g_{2*}^2.$$

In a certain region $g_{3*} \geq 7g_{2*}^2/8 + \varepsilon/6$ all these quantities are positive. Of course, this is just a necessary condition; still, we can assume that the surface of fixed points g_* contains a region of IR stability. If this is indeed so, the model may contain IR scaling with nonuniversal critical dimensions (*i.e.* there is a dependence on the parameters g_{2*} , and g_{3*}).

In dynamic models of the type (2.1) the critical exponents Δ_F of an arbitrary quantity F (a field or a parameter) is given by the following expression:

$$\Delta_F = d_F^\perp + d_F^\parallel \Delta_\parallel + d_F^\omega \Delta_\omega + \gamma_F^*, \quad \Delta_\omega = 2 - \gamma_\perp^*, \quad \Delta_\parallel = 1 + \gamma_\parallel^*/2. \quad (5.1)$$

In case at hand for $F = h$ we have $\gamma_h^* = 0$ and $\gamma_\perp^* = 0$ (the fields and the parameter ν_\perp are not renormalized). Relations (5.1) together with the table 1 yield the exact result $2\Delta_h = d - 1 + \Delta_\parallel - \Delta_\omega$; from (4.14) we find in the one-loop approximation that $\Delta_\parallel = 1 + a(2g_{3*} - g_{2*}^2)/4$, $\Delta_h = a(2g_{3*} - g_{2*}^2)/8$.

6. Conclusion

We applied to the modified model [14, 15] the standard field theoretic RG. It turned out that the model can be reformulated as a renormalizable field theoretic model with an infinite set of independent renormalization constants (thus, infinite set of coupling constants). Indeed, to construct renormalizable model it is necessarily to include infinitely many coupling constants, and the corresponding RG equations involve infinitely many β -functions. Despite this fact, it appears possible to derive the one-loop counterterm employing the method, earlier proposed in [27] for an isotropic model of surface roughening. The method yields a two-dimensional surface of fixed points which is likely to contain IR attractive region(s). Indeed, experimental results (see the discussion in [15]) indicate two wide ranges of roughening exponent value which might be explained by the existence of two different IR attractive regions.

As the model needs to contain infinite set of coupling constants to be renormalizable it seems that truncated models like [14, 15] or the one with odd $V(h)$ might not be suitable for the RG analysis. The naive approach of putting the corresponding coupling constants in $V(h)$ to zero in attempt to compare the results shows that in the case of the model [14, 15] there is no agreement.

To compare the critical exponents of those two models one has to identify $z_\perp = \Delta_\omega$, $\zeta_\perp = \Delta_\parallel$, $\alpha_\perp = \Delta_h$. Obvious calculations show that in the case of $\lambda_n = 0$ for all n but $n = 3$ the critical exponents are $\Delta_\omega = 2$, $\Delta_\parallel = 1 + (2-d)/6$, and $\Delta_h = (2-d)/12$. The last two values differ from the ones reported in [14, 15].

For odd $V(h)$ a two-dimensional surface of fixed points reduces to a curve. From the symmetry considerations, as well as from the explicit expression for the counterterm (4.8), it is clear that this case is renormalizable in itself. One can simply set all the odd couplings g_{2n+1} and the corresponding β functions in (4.13) equal to zero.

If the surface of fixed points does indeed contain IR attractive regions, than the model exhibits scaling behaviour. The corresponding scaling exponents turn out to be nonuniversal because of their dependence on the coordinates of specific fixed point on the surface (curve). Nonetheless, they satisfy certain exact relations.

In the further study, it would be interesting to investigate how the model behaves if there is a turbulent velocity field involved; for the isotropic case, see [28].

From a more theoretical point of view, it is desirable to write down the RG equations and to find the fixed point(s) directly in terms of the function $V(h)$, so that instead of infinitely many β functions for the infinite set of couplings g_n we would have the only $\beta(V)$ functional with the only functional argument $V(h)$; see the discussion in [33] for a general case.

This work remains for the future and is partly in progress.

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